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Calderón–Zygmund operators on Herz type Hardy spaces

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Abstract

We consider the boundedness of Calderón–Zygmund operators from $H\dot{K}_q^{\alpha,p}(R^n)$ to $h\dot{K}_q^{\alpha,p}(R^n)$, where $H\dot{K}_q^{\alpha,p}(R^n)$ is the Hardy space associated with the Herz space $\dot{K}_q^{\alpha,p}(R^n)$ and $h\dot{K}_q^{\alpha,p}(R^n)$ is the local version of $H\dot{K}_q^{\alpha,p}(R^n)$. We show Calderón's commutator is bounded from $H\dot{K}_q^{\alpha,p}$ to $h\dot{K}_q^{\alpha,p}$.

Keywords: Calderón–Zygmund operator; Hardy space; Herz space

1. Introduction

Consider the operator defined by

$$Tf(x) = \text{p.v.} \int_{R^n} K(x, y) f(y)\, dy,$$

where K is a Calderón–Zygmund kernel (see Section 2.4).

Alvarez and Milman [1,2] proved that if kernel $K(x, y)$ has some regularity then T is a bounded operator from $H^p(R^n)$ to $H^p(R^n)$ if $T^*1 = 0$. Komori [7] proved that if T^*1 belongs to Lipschitz class then T is a bounded operator from $H^p(R^n)$ to $h^p(R^n)$, where $h^p(R^n)$ is the local Hardy space defined by Goldberg [4].

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Lu and Yang [10] showed that T is a bounded operator from $H\dot{K}_q^{\alpha,p}(R^n)$ to $H\dot{K}_q^{\alpha,p}(R^n)$ if $T^*1 = 0$, where $H\dot{K}_q^{\alpha,p}(R^n)$ is the Hardy space associated with the Herz space $\dot{K}_q^{\alpha,p}(R^n)$ (see Section 2.2).

In this paper we define the space $h\dot{K}_q^{\alpha,p}(R^n)$ which is the local version of $H\dot{K}_q^{\alpha,p}$ and we shall show that if T^*1 belongs to Lipschitz class then T is a bounded operator from $H\dot{K}_q^{\alpha,p}(R^n)$ to $h\dot{K}_q^{\alpha,p}(R^n)$.

As a corollary of our theorem, we obtain the boundedness of Calderón's commutator.

2. Definitions and notations

The following notation is used: For a set $E \subset R^n$ we denote the Lebesgue measure of E by $|E|$. We denote the characteristic function of E by χ_E. We write a ball of radius r centered at x_0 by $B(x_0, r) = \{x;\ |x - x_0| < r\}$.

2.1. *Ordinary Hardy spaces*

First we shall define two maximal functions and ordinary Hardy spaces.

Let $\varphi \in \mathcal{S}$ be a fixed function such that $\mathrm{supp}(\varphi) \subset B(0,1)$, and $\int \varphi(x)\,dx \neq 0$. We define

$$f^{++}(x) = \sup_{t>0}\left|\int f(y)\varphi_t(x-y)\,dy\right|, \qquad f^{+}(x) = \sup_{1>t>0}\left|\int f(y)\varphi_t(x-y)\,dy\right|,$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Definition 1 (Fefferman–Stein's Hardy space [3]).

$$H^p(R^n) = \left\{f \in \mathcal{S}';\ \|f\|_{H^p} = \|f^{++}\|_{L^p} < \infty\right\}.$$

Definition 2 (Local Hardy space [4]).

$$h^p(R^n) = \left\{f \in \mathcal{S}';\ \|f\|_{h^p} = \|f^{+}\|_{L^p} < \infty\right\}.$$

Definition 3 (Lipschitz space).

$$\mathrm{Lip}_\varepsilon(R^n) = \left\{f;\ \|f\|_{\mathrm{Lip}_\varepsilon} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^\varepsilon} < \infty\right\} \quad \text{for } 0 < \varepsilon < 1.$$

Remark. $(H^p)^* = \mathrm{Lip}_{n(1/p-1)}$, where $n/(n+1) < p < 1$ (duality, see [3] or [9, p. 54]).

2.2. *Herz spaces*

Next we define the Herz spaces and the Hardy spaces associated with the Herz spaces (see [5,8,10]).

Let $0 < p \leqslant 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$.

Definition 4. The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in L^q_{\mathrm{loc}}\big(R^n \setminus \{0\}\big);\ \|f\|_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\int_{A_k} \left| f(x) \right|^q dx \right)^{p/q} \right\}^{1/p}$$

and $A_k = \{x \in R^n;\ 2^{k-1} \leqslant |x| < 2^k\}$.

Definition 5. The Hardy space $H\dot{K}_q^{\alpha,p}(R^n)$ associated with $\dot{K}_q^{\alpha,p}(R^n)$ is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in \mathcal{S}'(R^n);\ f^{++} \in \dot{K}_q^{\alpha,p}(R^n) \right\},$$

and we define $\|f\|_{H\dot{K}_q^{\alpha,p}} = \|f^{++}\|_{\dot{K}_q^{\alpha,p}}$.

Definition 6. The local Hardy space $h\dot{K}_q^{\alpha,p}(R^n)$ associated with $\dot{K}_q^{\alpha,p}(R^n)$ is defined by

$$h\dot{K}_q^{\alpha,p}(R^n) = \left\{ f \in \mathcal{S}'(R^n);\ f^{+} \in \dot{K}_q^{\alpha,p}(R^n) \right\},$$

and we define $\|f\|_{h\dot{K}_q^{\alpha,p}} = \|f^{+}\|_{\dot{K}_q^{\alpha,p}}$.

Remark. $\|f\|_{h\dot{K}_q^{\alpha,p}} \leqslant \|f\|_{H\dot{K}_q^{\alpha,p}}$.

2.3. Atom

We shall define atoms on the Hardy space. Let $n/(n+1) < p \leqslant 1$. First we define ordinary atom on $H^p(R^n)$.

Definition 7. Let $1 < q \leqslant \infty$. A function $a(x)$ is a (H^p, q)-atom centered at x_0 if there exists a ball $B(x_0, r)$ such that the following conditions are satisfied:

$$\operatorname{supp}(a) \subset B(x_0, r), \tag{1}$$

$$\|a\|_{L^q} \leqslant \left| B(x_0, r) \right|^{1/q-1/p}, \tag{2}$$

$$\int a(x)\, dx = 0. \tag{3}$$

Lemma 1 [9, p. 35]. *If a function $a(x)$ is a (H^p, q)-atom supported in a ball $B(x_0, r)$, then $\|a\|_{H^{p_1}} \leqslant C_{n,p_1,q} |B(x_0, r)|^{1/p_1-1/p}$, where $n/(n+1) < p_1 \leqslant 1$ and $C_{n,p_1,q}$ is a positive constant depending only on n, p_1, and q.*

Following Lu and Yang [10], we define atoms on $H\dot{K}_q^{\alpha,p}(R^n)$.

Definition 8. Let $1 < q < \infty$. A function $a(x)$ is a central $(H\dot{K}_q^{\alpha,p})$-atom if there exists a ball $B(0, r)$ such that the following conditions are satisfied:

$$\operatorname{supp}(a) \subset B(0, r), \tag{1$'$}$$

$$\|a\|_{L^q} \leqslant \left|B(0,r)\right|^{1/q-1/p}, \tag{2'}$$

$$\int a(x)\,dx = 0. \tag{3'}$$

Lemma 2 [10]. *If a function $a(x)$ is a central $(H\dot{K}_q^{\alpha,p})$-atom, then $\|a\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q}$.*

Lu and Yang [10] obtained the following atomic decomposition theorem for the Hardy space $H\dot{K}_q^{\alpha,p}(R^n)$.

Proposition. *If $f \in H\dot{K}_q^{\alpha,p}(R^n)$ then f can be represented as*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k, \quad \text{where } a_k \text{ is a central } (H\dot{K}_q^{\alpha,p})\text{-atom}$$

and $\sum_{k=1}^{\infty} |\lambda_k|^p \sim \|f\|^p_{H\dot{K}_q^{\alpha,p}}$.

2.4. Calderón–Zygmund operator

We shall define Calderón–Zygmund operator.

Definition 9. Let T be a bounded linear operator from $\mathcal{S}$ to $\mathcal{S}'$. T is called a standard operator if T satisfies the following conditions:

(i) T extends to a continuous operator on $L^2(R^n)$;
(ii) There exists a function $K(x,y)$ defined on $\{(x,y) \in R^n \times R^n;\ x \neq y\}$ which satisfies $|K(x,y)| \leqslant C/|x-y|^n$;
(iii) $(Tf, g) = \iint K(x,y) f(y) g(x)\,dy\,dx$ for $f, g \in \mathcal{S}$ with disjoint supports.

Definition 10. A standard operator T is called a δ-Calderón–Zygmund operator if $K(x,y)$ satisfies

$$\left|K(x,y) - K(x,z)\right| + \left|K(y,x) - K(z,x)\right| \leqslant C \frac{|y-z|^{\delta}}{|x-z|^{n+\delta}}$$

if $2|y-z| < |x-z|$ for some $0 < \delta \leqslant 1$.

Remark. If T is a δ-Calderón–Zygmund operator, then T is bounded on L^q, where $q > 1$ (see [6, p. 52]).

3. Theorems

Lu and Yang [10] obtained next result.

Theorem. *Let $1 < q < \infty$, $0 < \delta \leqslant 1$, and $n/(n+\delta) < p \leqslant 1$. If T is a δ-Calderón–Zygmund operator such that $T^*1 = 0$ then T is a bounded operator from $H\dot{K}_q^{\alpha,p}(R^n)$ to $H\dot{K}_q^{\alpha,p}(R^n)$.*

Remark. T^* is an adjoint operator of T. T and T^* are simultaneously δ-Calderón–Zygmund operators. For the definition of T^*1, see [11, p. 412].

We have the following

Theorem 1. *Let $1 < q < \infty$, $0 < \delta \leqslant 1$, $n/(n+\delta) < p \leqslant 1$, and $nq/(n+q\varepsilon) \leqslant p$. If T is a δ-Calderón–Zygmund operator such that $T^*1 \in \mathrm{Lip}_\varepsilon$ then T is a bounded operator from $H\dot{K}_q^{\alpha,p}(R^n)$ to $h\dot{K}_q^{\alpha,p}(R^n)$.*

Remark. We can consider the nonhomogeneous version of our theorem (see [10]).

As a corollary of Theorem 1 we obtain the boundedness of Calderón's commutator.

Definition 11. Calderón's commutator is defined by

$$T_b f(x) = \text{p.v.} \int_{R^1} \frac{b(x)-b(y)}{(x-y)^2} f(y)\,dy.$$

Theorem 2. *If $b' \in L^\infty \cap \mathrm{Lip}_\varepsilon$, then T_b is a bounded operator from $H\dot{K}_q^{\alpha,p}(R^1)$ to $h\dot{K}_q^{\alpha,p}(R^n)$, where $q/(1+q\varepsilon) \leqslant p \leqslant 1$.*

Proof. If $b' \in L^\infty$ then T_b is bounded on L^2 (see [11, p. 408]) and a 1-Calderón–Zygmund operator ($\delta = 1$). We can write $T_b^* 1(x) = -H(b')(x)$, where H is the Hilbert transform. Since H is bounded on Lip_ε (see [11, p. 214]), we have $T_b^* 1(x) \in \mathrm{Lip}_\varepsilon$. By Theorem 1 we obtain the desired result. $\square$

4. Molecule

In this section we shall define block and molecule on $h\dot{K}_q^{\alpha,p}(R^n)$ and prove some properties. Let $n/(n+1) < p \leqslant 1$.

Definition 12 [10]. Let $1 < q < \infty$. We say a function $a(x)$ is a central $(h\dot{K}_q^{\alpha,p})$-block if a satisfies $(1')$ and $(2')$ in Definition 8.

The following Lemma 3 is essentially proved in [10, p. 108].

Lemma 3. *Let $1 < q < \infty$. If a function $a(x)$ is a central $(h\dot{K}_q^{\alpha,p})$-block such that $\mathrm{supp}(a) \subset B(0,r)$, where $r \geqslant 1$, then we have $\|a\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q}$.*

Proof. We assume $2^{k_0} \leqslant r < 2^{k_0+1}$ for some $k \in \mathbb{Z}$. Note that $a^+(x) = 0$ if $x \notin B(0,2r)$. By the L^q boundedness of the Hardy–Littlewood maximal function, we have

$$\sum_{k=-\infty}^{k_0+2} 2^{k\alpha p}\left(\int_{A_k} \left|a^+(x)\right|^q dx\right)^{p/q} \leqslant C_{n,q} \sum_{k=-\infty}^{k_0+2} 2^{k\alpha p}\|a\|_{L^q}^p$$
$$\leqslant C_{n,p,q} 2^{k_0\alpha p}\left|B(0,r)\right|^{p(1/q-1/p)} \leqslant C_{n,p,q}. \qquad \square$$

Lemma 4. *Let $1 < q < \infty$. If a function $a(x)$ satisfies $(1')$, $(2')$, and*

$$\left|\int a(x)\,dx\right| \leqslant \left|B(0,r)\right|^{1-1/q} \tag{3'}$$

for some $0 < r < 2$, then we have $\|a\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q}$.

Proof. We write

$$a(x) = \big(a(x) - a_B\big)\chi_B(x) + a_B\chi_B(x) = a_1(x) + a_2(x),$$

where $B = B(0,r)$ and $a_B = (1/|B|)\int_B a(y)\,dy$.

We have

$$\left(\int \left|a_1(x)\right|^q dx\right)^{1/q} \leqslant 2\|a\|_{L^q} \leqslant 2\left|B(0,r)\right|^{1/q-1/p}.$$

Therefore $a_1/2$ is a central $(H\dot{K}_q^{\alpha,p})$-atom and we have $\|a_1\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q}$ by Lemma 2.

Then $\operatorname{supp}(a_2) \subset B(0,2)$ and

$$\|a_2\|_{L^q} \leqslant |a_B|\left|B(0,r)\right|^{1/q} \leqslant C_{n,p,q}\left|B(0,2)\right|^{1/q-1/p}.$$

So a_2 is a constant multiple of a central $(h\dot{K}_q^{\alpha,p})$-block. By Lemma 3 we have $\|a_2\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q}$. $\square$

Definition 13. Let $\delta > 0$. We say a function $M(x)$ is a large (p,q,δ)-molecule if M satisfies

$$\left(\int_{|x|<2r} \left|M(x)\right|^q dx\right)^{1/q} \leqslant \left|B(0,r)\right|^{(1/q-1/p)}, \tag{M$_1$}$$

$$\left|M(x)\right| \leqslant \frac{|B(0,r)|^{1+\delta/n-1/p}}{|x|^{n+\delta}}, \quad \text{where } |x| \geqslant 2r, \tag{M$_2$}$$

for some $r \geqslant 1$.

We say a function $M(x)$ is a small (p,q,δ)-molecule if M satisfies (M$_1$), (M$_2$), and

$$\left|\int M(x)\,dx\right| \leqslant \left|B(0,r)\right|^{1-1/q} \tag{M$_3$}$$

for some $0 < r < 1$.

Remark. For the definition of molecule on $H\dot{K}_q^{\alpha,p}$, see [10, p. 116].

Lemma 5. *Let $n/(n+\delta) < p \leqslant 1 < q < \infty$. If a function $M(x)$ is a large or small (p,q,δ)-molecule, then we have $\|M\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$.*

Proof. Let $E_0 = \{x;\ |x| < 2r\}$ and $E_k = \{x;\ 2^k r \leqslant |x| < 2^{k+1} r\}$, $k = 1, 2, 3, \ldots,$ and let

$$\chi_k(x) = \chi_{E_k}(x), \qquad \tilde{\chi}_k(x) = \frac{1}{|E_k|}\chi_{E_k}(x),$$
$$m_k = \frac{1}{|E_k|}\int_{E_k} M(y)\,dy, \qquad \tilde{m}_k = \int_{E_k} M(y)\,dy,$$

and $M_k(x) = (M(x) - m_k)\chi_k(x)$.

We write

$$M(x) = \sum_{k=0}^{\infty} M_k(x) + \sum_{k=0}^{\infty} m_k\chi_k(x) = \sum_{k=0}^{\infty} M_k(x) + \sum_{k=0}^{\infty} \tilde{m}_k\tilde{\chi}_k(x).$$

Let $N_k = \sum_{j=k}^{\infty} \tilde{m}_j$ and we write

$$M(x) = \sum_{k=0}^{\infty} M_k(x) + \sum_{k=1}^{\infty} N_k\big(\tilde{\chi}_k(x) - \tilde{\chi}_{k-1}(x)\big) + N_0\tilde{\chi}_0(x) = I + II + III.$$

We shall show $\|I\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$, $\|II\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$, and $\|III\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$.

First we estimate I. It is clear that $\operatorname{supp}(M_k) \subset B(0, 2^{k+1}r)$, $\int M_k(x)\,dx = 0$.

By using condition (M_1), we have

$$\left(\int \big|M_0(x)\big|^q\,dx\right)^{1/q} \leqslant 2\big|B(x_0, 2r)\big|^{1/q-1/p}.$$

Therefore we obtain $\|M_0\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q}$ by Lemma 2.

Using condition (M_2), we have

$$\big|M_k(x)\big| \leqslant (2^k r)^{-n-\delta}\big|B(0,r)\big|^{1+\delta/n-1/p} \leqslant C_n\, 2^{(-n-\delta+n/p)k}\big|B(0, 2^{k+1}r)\big|^{-1/p}.$$

By Lemma 2 we have $\|M_k\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q}\, 2^{(-n-\delta+n/p)k}$.

Since $p > n/(n+\delta)$, we obtain $\sum_{k=0}^{\infty} \|M_k\|_{H\dot{K}_q^{\alpha,p}}^p \leqslant C_{n,p,q,\delta}$ and $\|I\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$.

Next we estimate II. Let $A_k(x) = N_k(\tilde{\chi}_k(x) - \tilde{\chi}_{k-1}(x))$. It is clear that $\operatorname{supp}(A_k) \subset B(0, 2^{k+1}r)$, $\int A_k(x)\,dx = 0$. By the same estimate as I we have

$$\begin{aligned}\|A_k\|_{L^\infty} &\leqslant C_n(2^k r)^{-n}\int_{2^{k-1}r\leqslant|x|<2^{k+1}r}\big|M(x)\big|\,dx\\ &\leqslant C_n 2^{(-n-\delta+n/p)k}\big|B(0, 2^{k+1}r)\big|^{-1/p}.\end{aligned}$$

So we obtain $\sum_{j=k}^{\infty} \|A_k\|_{H\dot{K}_q^{\alpha,p}}^p \leqslant C_{n,p,q,\delta}$ and $\|II\|_{H\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$.

Finally we estimate III. It is clear that $\operatorname{supp}(N_0\tilde{\chi}_0) \subset B(0, 2r)$. By the same estimate as I, we have

$$
\begin{aligned}
\|N_0\tilde{\chi}_0\|_{L^\infty} &\leqslant \frac{1}{|B(0,2r)|}\int \bigl|M(x)\bigr|\,dx \\
&\leqslant \frac{1}{|B(0,2r)|}\left(\int\limits_{|x|<2r}\bigl|M(x)\bigr|\,dx + \int\limits_{|x|\geqslant 2r}\bigl|M(x)\bigr|\,dx\right) \\
&\leqslant C_{n,p,q,\delta}\bigl|B(0,2r)\bigr|^{-1/p}.
\end{aligned} \tag{4}
$$

If $r \geqslant 1$, by (4) and Lemma 3 we have $\|N_0\tilde{\chi}_0\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$.

If $r < 1$, using condition (M_3), we have

$$
\left|\int N_0\tilde{\chi}_0(x)\,dx\right| = \left|\int M(x)\,dx\right| \leqslant C_{n,q}\bigl|B(0,2r)\bigr|^{1-1/q}. \tag{5}
$$

By (4), (5), and Lemma 4 we have $\|N_0\tilde{\chi}_0\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$.

So we obtain $\|III\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$. □

5. Proof of Theorem 1

By the atomic decomposition of $H\dot{K}_q^{\alpha,p}(R^n)$, it suffices to show that there exists $C_{n,p,q,\varepsilon,\delta,T} > 0$ such that $\|Ta\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\varepsilon,\delta,T}$ for every central $(H\dot{K}_q^{\alpha,p})$-atom a, where $C_{n,p,q,\varepsilon,\delta,T}$ is a positive constant depending only on n, p, q, ε, δ, and $\|T\|_{\mathrm{Lip}_\varepsilon}$.

We assume a $(H\dot{K}_q^{\alpha,p})$-atom a is supported in $B(0,r)$. We shall show that if $r \geqslant 1$ then $Ta(x)$ is a constant multiple of a large (p,q,δ)-molecule, and if $r < 1$ then $Ta(x)$ is a constant multiple of a small (p,q,δ)-molecule.

We have to check that if $r \geqslant 1$ then Ta satisfies (M_1) and (M_2), and if $r < 1$ then Ta satisfies three conditions of Definition 13.

Since T is bounded on L^q [6, p. 52], we have

$$
\left(\int\limits_{|x|<2r}\bigl|Ta(x)\bigr|^q\,dx\right)^{1/q} \leqslant C_{n,q}\|a\|_{L^q} \leqslant C_{n,q}\bigl|B(0,r)\bigr|^{1/q-1/p}. \tag{6}
$$

If $|x| \geqslant 2r$, we have

$$
\bigl|Ta(x)\bigr| = \left|\int \bigl(K(x,y) - K(x,x_0)\bigr)a(y)\,dy\right| \leqslant C_n\frac{|B(0,r)|^{1+\delta/n-1/p}}{|x|^{n+\delta}}. \tag{7}
$$

If $r \geqslant 1$, by (6), (7), and Lemma 5, we have $\|Ta\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\delta}$.

If $r < 1$, by the duality of $H^{n/(n+\varepsilon)}$ and Lip_ε and Lemma 1, we have

$$
\begin{aligned}
\left|\int Ta(x)\,dx\right| &= \bigl|(Ta,1)\bigr| = \bigl|(a,T^*1)\bigr| \leqslant C_n\|a\|_{H^{n/(n+\varepsilon)}}\|T^*1\|_{\mathrm{Lip}_\varepsilon} \\
&\leqslant C_n\|T^*1\|_{\mathrm{Lip}_\varepsilon}\bigl|B(0,r)\bigr|^{(n+\varepsilon)/n-1/p} \\
&\leqslant C_n\|T^*1\|_{\mathrm{Lip}_\varepsilon}\bigl|B(0,r)\bigr|^{1-1/q},
\end{aligned} \tag{8}
$$

because $p \geqslant nq/(n+q\varepsilon)$.

By (6)–(8) and Lemma 5, we obtain $\|Ta\|_{h\dot{K}_q^{\alpha,p}} \leqslant C_{n,p,q,\varepsilon,\delta,T}$. □

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